

Static Optimization with MATLAB

Day 4: Comparative Statics and Value Functions

GSEFM Pre-Semester Course

Luca Schmitz

Goethe University Frankfurt

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What is Comparative Statics?

⇒ **What is the effect of a change in an exogenous parameter on an endogenous variable?**

In economic applications, one may be interested to see how

- **optimal decisions change**, e.g., how
 - the optimal mix of a firm's inputs changes when the price of its inputs changes,
 - the expenditure habits change when the price of a good changes
- **value functions change**, e.g., how
 - a firm's profit changes when it has more resources to spend on production,
 - a consumer's utility changes when they have a higher income to spend on consumption

The Response of Optimal Decisions

Definition (Problem Setting)

We want to determine the signs of the partial derivatives of the decision function

$$x(\alpha) = \arg \max_{x \in M(\alpha)} f(x, \alpha)$$

with respect to the parameters $\frac{\partial x_i^*}{\partial \alpha_k}$.

⇒ will make use of the **Implicit Function Theorem (IFT)**.

Exercise

An agent consumes two goods, x_1 and x_2 , with prices p_1 and p_2 , respectively. Her utility function is of the form $U(x_1, x_2) = \alpha(x_1^\alpha + x_2^\alpha)$, with $\alpha < 1$ and her income is y .

1. Verify that $U()$ is strictly concave.
2. Derive the demand function of the agent.
3. In what direction does the demand for good 1 change if there is an increase in the price of good 2?

Exercise Solution I

1. Strict concavity of $U()$:

$$U_1 = \frac{\partial U}{\partial x_1} = \alpha^2 x_1^{\alpha-1}, \quad U_2 = \frac{\partial U}{\partial x_2} = \alpha^2 x_2^{\alpha-1}$$

$$U_{11} = \frac{\partial^2 U}{\partial x_1^2} = \alpha^2(\alpha-1)x_1^{\alpha-2}, \quad U_{12} = 0$$

$$U_{21} = 0, \quad U_{22} = \alpha^2(\alpha-1)x_2^{\alpha-2}$$

The Hessian matrix is therefore

$$H = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} \alpha^2(\alpha-1)x_1^{\alpha-2} & 0 \\ 0 & \alpha^2(\alpha-1)x_2^{\alpha-2} \end{bmatrix}$$

Exercise Solution II

The leading principal minors are (provided x_1 and x_2 are strictly positive)

$$d_1 = \alpha^2(\alpha - 1)x_1^{\alpha-2} < 0$$

$$d_2 = \alpha^2(\alpha - 1)x_1^{\alpha-2}\alpha^2(\alpha - 1)x_2^{\alpha-2} = \alpha^4(\alpha - 1)^2x_1^{\alpha-2}x_2^{\alpha-2} > 0$$

Hence, H is negative definite and the sufficient conditions for a strict local maximum are satisfied.

Exercise Solution III

2. Demand functions:

Firstly, write the Lagrangian function of the problem:

$$\mathcal{L}(x_1, x_2, \lambda) = \alpha(x_1^\alpha + x_2^\alpha) - \lambda[p_1x_1 + p_2x_2 - y]$$

First-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1}(x_1^*, x_2^*, \lambda^*) = \alpha^2 x_1^{*\alpha-1} - \lambda^* p_1 = 0 \Rightarrow \lambda^* = \frac{\alpha^2}{p_1 x_1^{*1-\alpha}}$$

$$\frac{\partial \mathcal{L}}{\partial x_2}(x_1^*, x_2^*, \lambda^*) = \alpha^2 x_2^{*\alpha-1} - \lambda^* p_2 = 0 \Rightarrow \lambda^* = \frac{\alpha^2}{p_2 x_2^{*1-\alpha}}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x_1^*, x_2^*, \lambda^*) = p_1 x_1^* + p_2 x_2^* - y = 0$$

Exercise Solution IV

Equating expressions for λ^* :

$$\frac{\alpha^2}{p_1 x_1^{*1-\alpha}} = \frac{\alpha^2}{p_2 x_2^{*1-\alpha}} \Rightarrow \frac{p_1}{p_2} = \left(\frac{x_2^*}{x_1^*} \right)^{1-\alpha} \Rightarrow x_2^* = \left(\frac{p_1}{p_2} \right)^{\frac{1}{1-\alpha}} x_1^*$$

Plug into 3rd FOC (i.e. constraint):

$$p_1 x_1^* + p_2 \left(\frac{p_1}{p_2} \right)^{\frac{1}{1-\alpha}} x_1^* = y$$
$$\Leftrightarrow x_1^* = x_1(p_1, p_2, y) = \frac{y}{p_1 + p_2 \left(\frac{p_1}{p_2} \right)^{\frac{1}{1-\alpha}}} = \frac{y}{p_1 \left(1 + \left(\frac{p_1}{p_2} \right)^{\frac{\alpha}{1-\alpha}} \right)}$$

Exercise Solution V

Use solution for $x_1^* = x_1(p_1, p_2, y)$ to get $x_2^* = x_2(p_1, p_2, y)$ and $\lambda^* = \lambda(p_1, p_2, y)$

$$x_2^* = x_2(p_1, p_2, y) = \frac{y}{p_2 \left(1 + \left(\frac{p_1}{p_2} \right)^{\frac{\alpha}{1-\alpha}} \right)}$$
$$\lambda^* = \lambda(p_1, p_2, y) = \frac{\alpha^2 p_2^{-\alpha} \left(1 + \left(\frac{p_1}{p_2} \right)^{\frac{\alpha}{1-\alpha}} \right)^{1-\alpha}}{y^{1-\alpha}}$$

Exercise Solution VI

3. Effect of change in p_2 on x_1^* :

Differentiating $x_1(p_1, p_2, y)$ w.r.t. p_2 :

$$\frac{\partial x_1(p_1, p_2, y)}{\partial p_2} = \frac{y \frac{\alpha}{1-\alpha} p_1^{\frac{\alpha}{1-\alpha}} p_2^{-\frac{\alpha}{1-\alpha}-1}}{p_1 \left(1 + \left(\frac{p_1}{p_2} \right)^{\frac{\alpha}{1-\alpha}} \right)^2}$$

Because $1 - \alpha > 0$, the sign of the derivative is only depending on the sign of α .

- $\alpha > 0 \Rightarrow \frac{\partial x_1(p_1, p_2, y)}{\partial p_2} > 0 \Rightarrow$ The two goods are substitutes
- $\alpha < 0 \Rightarrow \frac{\partial x_1(p_1, p_2, y)}{\partial p_2} < 0 \Rightarrow$ The two goods are complements

Implicit Function Theorem I

Idea: The IFT describes a general way how to differentiate a function g that is defined implicitly through an system of equations,

$$F(x, g(x)) = 0.$$

$\Rightarrow$ In our simplest (unconstrained case), we try to obtain the derivative $D_\alpha x$ of the optimal solution $x(\alpha)$ which is defined through the first-order conditions (FOCs) of the maximization problem:

$$D_x f(x(\alpha), \alpha) \equiv \mathbf{0}.$$

The derivative of $D_\alpha x$ is given by:

$$D_x(\alpha^0) = -[D_x^2 f(x^0, \alpha^0)]^{-1} D_{x\alpha} f(x^0, \alpha^0),$$

Implicit Function Theorem II

- We need to differentiate the FOCs which involve already first derivatives of f and g , therefore, they both must be twice continuously differentiable functions.
- The solution to the programming problem must be a regular maximum (i.e. satisfying the conditions for a strict local maximum). This ensures the matrix obtained by differentiating the system of FOCs w.r.t. x is invertible ($|H| = |D_x^2 f(x^0, \alpha^0)| \neq 0$)

Unconstrained Problem: Implicit Function Theorem I

Assume that the conditions for the IFT hold at (x^0, α^0) for an unconstrained problem

$\Rightarrow$ The solution mapping for the maximization problem, $x^* = x(\alpha)$, is locally a well-defined and smooth function. Substituting $x()$ into the first-order conditions, we obtain the identity

$$D_x f(x(\alpha), \alpha) \equiv \mathbf{0}.$$

Differentiating with respect to α and evaluated at the optimum:

$$D_x^2 f(x^0, \alpha^0) D_x(\alpha^0) + D_{x\alpha} f(x^0, \alpha^0) = \mathbf{0}$$

and solving for $D_x(\alpha^0)$ yields

$$D_x(\alpha^0) = -[D_x^2 f(x^0, \alpha^0)]^{-1} D_{x\alpha} f(x^0, \alpha^0)$$

Unconstrained Problem: IFT II

Using Cramer's rule to solve the linear system

$$D_x^2 f(x^0, \alpha^0) D_x(\alpha^0) = -D_{x\alpha} f(x^0, \alpha^0),$$

we can obtain each of the components of the solution vector as the ratio of two determinants

$$\frac{\partial x_i^*}{\partial \alpha_k} = -\frac{|H_i|}{|H|},$$

where $|H_i|$ is the determinant of the matrix obtained by replacing the i th column of the Hessian with the k th column of $D_{x\alpha} f(x^0, \alpha^0)$, that is, the partials $\frac{\partial^2 f(x^0, \alpha^0)}{\partial x_j \partial \alpha_k}$ for $j = 1, \dots, n$.

$\Rightarrow$ There is also a Theorem for constrained problems, but we will not cover it here.

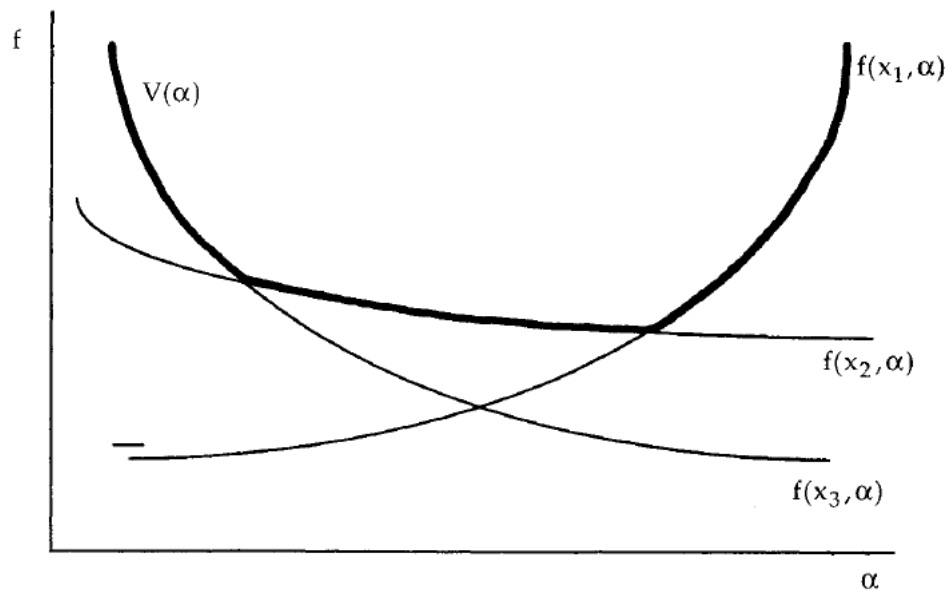
Maximum Value Function I

The value function $V : \omega \rightarrow \mathbb{R}$ for a maximization problem gives the maximum attainable value of the objective function for each value of the parameters:

$$\begin{aligned} V(\alpha) &= \max_{x \in \mathcal{S}} f(x; \alpha) \\ &= f(x(\alpha), \alpha) \end{aligned}$$

Graphically, maximum-value function corresponds to the upper "envelope" of the family of curves in α for each x .

Maximum Value Function II



Envelope Theorems: Motivation

- Allow us to relate the derivatives of the value function to those of the underlying objective and constraint functions.
- Often more convenient than the traditional route via the IFT and implicit differentiation of the first-order conditions.

Envelope Theorem (unconstrained maximization) I

Theorem (Envelope Theorem I)

Let $f(x, \alpha)$ be a twice continuously differentiable function, and let x^0 be a regular maximum (local maximum, i.e. stationary point and evaluated Hessian is negative definite) of f for α^0 . Then

$$V(\alpha) = \max_x f(x; \alpha) = f(x(\alpha), \alpha)$$

is differentiable at α^0 and $DV(\alpha^0) = D_\alpha f(x^0, \alpha^0)$.

Envelope Theorem (unconstrained maximization) II

Sketch of the proof:

- Differentiating $V(\alpha)$ w.r.t. α :

$$DV(\alpha^0) = \underbrace{D_x f(x^0; \alpha^0) D\alpha(\alpha^0)}_{\text{indirect effect, }=0} + \underbrace{D_\alpha f(x^0, \alpha^0)}_{\text{direct effect}}$$

- The FOCs of the problem, $D_x f(x^0; \alpha^0) = \mathbf{0}$ ensure that the marginal gain from small changes in the values of the choice variables will be zero when we start from the optimum.
 $\Rightarrow$ only need to consider direct effect.

Envelope Theorem (Lagrange Problem)

Theorem (Envelope Theorem II)

Let

$$V(\alpha) = \max_x f(x; \alpha), \text{ s.t. } g(x, \alpha) = 0$$

where f and g are twice continuously differentiable. If x^0 is a regular solution (strict local maximum) of this problem for α^0 , then V is differentiable at α^0 and

$$DV(\alpha^0) = D_\alpha \mathcal{L}(x^0, \alpha^0) = D_\alpha f(x^0, \alpha^0) - \lambda^{0'} D_\alpha g(x^0, \alpha^0)$$

We do not consider Kuhn-Tucker here because it is problematic that the decision rule may not be differentiable at the points where a regime change occurs. The same is true for the value function.

Envelope Theorem (Lagrange Problem) "Proof" I

Differentiating $V(\alpha) := f(x(\alpha), \alpha)$ w.r.t α :

$$DV(\alpha^0) = D_\alpha f(x^0, \alpha^0) + D_x f(x^0; \alpha^0) D\mathbf{x}(\alpha^0)$$

From the FOC of the problem we know

$$\begin{aligned} D_x \mathcal{L}(x, \alpha^0) &= D_x f(x, \alpha^0) - \lambda' D_x g(x, \alpha^0) = \mathbf{0} \\ &\Leftrightarrow D_x f(x, \alpha^0) = \lambda' D_x g(x, \alpha^0) \end{aligned}$$

Substituting in the above:

$$DV(\alpha^0) = D_\alpha f(x^0, \alpha^0) + \lambda' D_x g(x^0, \alpha^0) D\mathbf{x}(\alpha^0)$$

Envelope Theorem (Lagrange Problem) "Proof" II

We also have the second set of FOCs of the problem:

$$D_{\lambda}\mathcal{L}(x, \alpha^0) = g(x, \alpha^0) = \mathbf{0}$$

Substituting the decision rule $x(\alpha)$ in and differentiating w.r.t. α yields:

$$D_x g(x^0, \alpha^0) D_x(\alpha^0) - D_{\alpha} g(x^0, \alpha^0) = \mathbf{0}$$

With this result,

$$DV(\alpha^0) = D_{\alpha} f(x^0, \alpha^0) - \lambda' D_{\alpha} g(x^0, \alpha^0)$$

Economic Application

Consider a following constrained maximization problem with value function

$$V(p, y) = \max_x U(x; p, y) \quad s.t. \quad px = y$$

Let $x^* = x(p, y) > 0$ be a solution to this problem.

We want to know what is the change in the agent's optimized utility if her income changes

$$\frac{dV(p, y)}{dy}?$$

or if the price of a certain good changes:

$$\frac{dV(p, y)}{dp_j}?$$

Economic Application: The Problem

First of all, recall the Lagrangian of the problem

$$\mathcal{L}(x, \lambda; p, y) = U(x) - \lambda[px - y]$$

with FOCs:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x}(x^*, \lambda^*; p^0, y^0) &= \frac{\partial U}{\partial x}(x^*) - \lambda^* p^0 = \mathbf{0} \\ \frac{\partial \mathcal{L}}{\partial \lambda}(x^*, \lambda^*; p^0, y^0) &= p^0 x^* - y^0 = \mathbf{0}\end{aligned}$$

where $x^* = x(p, y)$ and $\lambda^* = \lambda(p, y)$ are functions of the parameters p and y . Plugging in the policy functions for x and λ , the value function of the problem can be restated as:

$$V(p^0, y^0) = U(x(p^0, y^0)) - \lambda(p^0, y^0)[p^0 x(p^0, y^0) - y^0]$$

Economic Application: Income change

Now, differentiate the value function w.r.t y :

$$\frac{dV}{dy} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial y} - p \left[\frac{\partial \lambda}{\partial y} x + \lambda \frac{\partial x}{\partial y} \right] + \left[\frac{\partial \lambda}{\partial y} y + \lambda \right]$$

This we can rearrange to

$$\begin{aligned} \frac{dV}{dy} &= \frac{\partial x}{\partial y} \underbrace{\left[\frac{\partial U}{\partial x} - p\lambda \right]}_{=0 \text{ according to FOC 1}} + \frac{\partial \lambda}{\partial y} \underbrace{[-px + y]}_{=0 \text{ according to constraint}} + \lambda \\ &= \lambda \end{aligned}$$

The marginal utility of an extra unit of income y is the shadow price $\lambda(p, y)$. A change in income y only affects the optimal value of U through its direct effect on the binding constraint, not through an indirect effect of a change in the composition of the optimal consumption bundle x^* .

Economic Application: Price change I

Differentiate the value function w.r.t p_j :

$$\frac{dV}{dp_j} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial p_j} - \left[\frac{\partial \lambda}{\partial p_j} (p_j x_j + \sum p_{-j} x_{-j}) + \lambda \left[x_j + p_j \frac{\partial x}{\partial p_j} \right] \right] + \frac{\partial \lambda}{\partial p_j} y$$

Rearranging:

$$\begin{aligned} \frac{dV}{dp_j} &= \frac{\partial x}{\partial p_j} \underbrace{\left[\frac{\partial U}{\partial x} - \lambda p_j \right]}_{=0 \text{ according to FOC 1}} + \frac{\partial \lambda}{\partial p_j} \underbrace{\left[y - (p_j x_j + \sum p_{-j} x_{-j}) \right]}_{=0 \text{ according to constraint}} - \lambda x_j \\ &= -\lambda x_j \end{aligned}$$

Same result when applying the Envelope Theorem directly.

Economic Application: Price change II

Can substitute result from income change from above:

$$\frac{dV(p, y)}{dp_j} = - \frac{dV(p, y)}{dy} x_j$$
$$x_j = - \frac{\frac{dV(p, y)}{dp_j}}{\frac{dV(p, y)}{dy}}$$

Which is known as **Roy's Identity**.

Knowing the indirect utility function $V(p, y)$, we can therefore use Roy's identity to recover Marshallian demand.

Profit Maximization

Consider the profit function

$$\Pi(p, w) = \max_x \pi(x; p, w) = f(x)p - xw$$

where p denotes the output price of one unit of output produced with the input x according to the production function $f(x)$ and w the input price of one unit of the input good..

Suppose $x^* = x(p, w)$ solves the maximization problem.

Prove **Hotelling's Lemma**:

$$\frac{d\Pi(p, w)}{dp} = f(x(p, w))$$

Profit Maximization Solution

$x(p, w)$ being a solution means that it has to satisfy the necessary first-order condition:

$$\frac{\partial \pi}{\partial x} = \frac{\partial f}{\partial x}(x(p, w))p - w = 0$$

Given the optimal solution, the value function is equal to

$$\Pi(p, w) = f(x(p, w))p - x(p, w)w$$

Differentiation with respect to p gives

$$\begin{aligned}\frac{d\Pi}{dp} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial p} p + f(x(p, w)) - \frac{\partial x}{\partial p} w \\ &= \frac{\partial x}{\partial p} \underbrace{\left(\frac{\partial f}{\partial x} p - w \right)}_{0 \text{ due to FOC above}} + f(x(p, w)) \\ &= f(x(p, w))\end{aligned}$$

Consumption-Savings Exercise I

An agent lives for 2 periods and has an endowment of 1 unit of a homogeneous consumption good in the first period, and γ units in the second period. His utility function over both periods is given by

$$U(c_1, c_2) = \ln(c_1) + \ln(c_2)$$

The agent can store part of his endowment for consumption in the second period and can get an interest-free loan of up to κ units (i.e. $R = 1 + r = 1$, $s \geq -\kappa$)

1. Calculate the agent's saving function, ignoring the constraint $s \geq -\kappa$.
2. for what combinations of parameters (γ, κ) will the constraint be binding? In what regions will we have an interior vs. a corner solution? Write the agent's savings function, taking into account the constraint.

Consumption-Savings Exercise II

3. Write the maximum-value function for the problem as a function of γ , $V(\gamma)$. Verify that $V(\gamma)$ is continuous at the point at which there is a regime change (i.e. as we go from an interior solution to one in which the constraint is binding). Is the value function differentiable at this point?

Consumption-Savings Exercise Solution I

1. Optimization without constraint

The period budget constraints are

$$c_1 + s = 1, \quad c_2 = \gamma + s$$

Substituting into the objective function, the agent solves

$$\max_s U(s) = \ln(1 - s) + \ln(\gamma + s)$$

The FOC is

$$\frac{\partial U}{\partial s}(s^*) = -\frac{1}{1 - s^*} + \frac{1}{\gamma + s^*} = 0$$

Solving for s^* gives

$$s^* = s(\gamma) = \frac{1 - \gamma}{2}$$

Strict concavity of the objective function guarantees a unique maximum.

Consumption-Savings Exercise Solution II

2. Constraint optimization

2 cases are possible:

- If $s(\gamma, \kappa) = \frac{1-\gamma}{2} \geq -\kappa$, the agent does not violate the constraint and the solution above prevails for the constrained equilibrium
- if $s(\gamma, \kappa) = \frac{1-\gamma}{2} \leq -\kappa$, the agent would like to borrow more than is allowed. Therefore, he will stick with the largest feasible amount and we have the corner solution of $s(\gamma, \kappa) = -\kappa$

The constraint will therefore be binding iff

$$s(\gamma, \kappa) = \frac{1-\gamma}{2} \leq -\kappa \quad \Rightarrow \quad \gamma \geq 1 + 2\kappa$$

Consumption-Savings Exercise Solution III

3. Value function and its properties

To recover the value function, substitute the optimal solution into the objective function:

$$V(\gamma, \kappa) = \ln(1 - s(\gamma, \kappa)) + \ln(\gamma + s(\gamma, \kappa))$$

As $s(\gamma, \kappa)$ has two different "segments", so will $V()$:

- if the constraint is not binding ($\gamma < 1 + 2\kappa$):

$$V^n(\gamma) = \ln\left(1 - \frac{1 - \gamma}{2}\right) + \ln\left(\gamma + \frac{1 - \gamma}{2}\right) = 2 \ln\left(\frac{1 + \gamma}{2}\right)$$

- if the constraint is binding ($\gamma \geq 1 + 2\kappa$):

$$V^b(\gamma, \kappa) = \ln(1 + \kappa) + \ln(\gamma - \kappa)$$

Consumption-Savings Exercise Solution IV

Continuity at the point of regime-switch: $\gamma^s = 1 + 2\kappa$:

$$\begin{aligned} V^n(\gamma^s) &= 2 \ln \left(\frac{1 + (1 + 2\kappa)}{2} \right) = 2 \ln(1 + \kappa) \\ V^b(\gamma^s, \kappa) &= \ln(1 + \kappa) + \ln(1 + 2\kappa - \kappa) = 2 \ln(1 + \kappa) \\ \Rightarrow V^n(\gamma^s) &= V^b(\gamma^s, \kappa) \end{aligned}$$

Value function is continuous.

Consumption-Savings Exercise Solution V

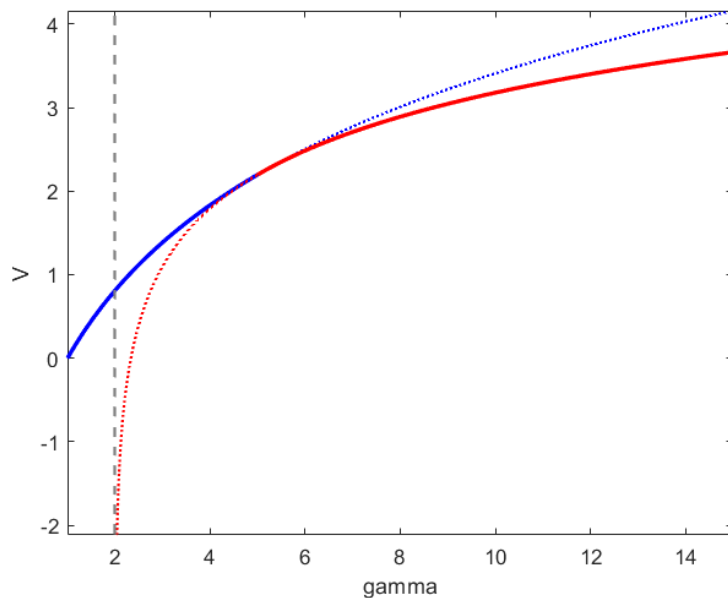
Differentiability at the point of regime-switch:

$$\begin{aligned}\frac{dV^n(\gamma^s)}{d\gamma} &= \frac{2}{1 + \gamma^0} \\ \frac{dV^b(\gamma^s, \kappa)}{d\gamma} &= \frac{1}{\gamma^0 - \kappa} = \frac{1}{\gamma^0 - \frac{\gamma^0 - 1}{2}} = \frac{2}{1 + \gamma^0} \\ \Rightarrow \frac{dV^n(\gamma^s)}{d\gamma} &= \frac{dV^b(\gamma^s, \kappa)}{d\gamma}\end{aligned}$$

Derivatives at γ^0 coincide $\Rightarrow V$ is differentiable at γ^s .

Consumption-Savings Exercise Solution VI

Value function for $\kappa = 2$ with turning point at $\gamma = 1 + 2 \cdot 2 = 5$



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